### A7 WHERE APPROPRIATE, INTERPRET SIMPLE EXPRESSIONS AS FUNCTIONS WITH INPUTS AND OUTPUTS; INTERPRET THE REVERSE PROCESS AS THE 'INVERSE FUNCTION'; INTERPRET THE SUCCESSION OF TWO FUNCTIONS AS A 'COMPOSITE FUNCTION' (THE USE OF FORMAL FUNCTION NOTATION IS EXPECTED) (higher tier)

In GCSE Mathematics equations are written as shown below:

$$y = 3x + 4 \qquad \qquad y = x^2 + 5$$

Sometimes a different notation is used which is called **function notation**.

We often use the letters f and g and we write the above equations as

f(x) = 3x + 4  $g(x) = x^2 + 5$ 

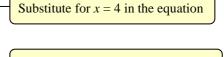
## EXAMPLE 1

Using the equation y = 3x + 4, find the value of y if

(a) x = 4 (b) x = -6

(a) 
$$y = 3(4) + 4 = 12 + 4 = 16$$

(b) 
$$y = 3(-6) + 4 = -18 + 4 = -14$$



— Substitute for x = -6 in the equation

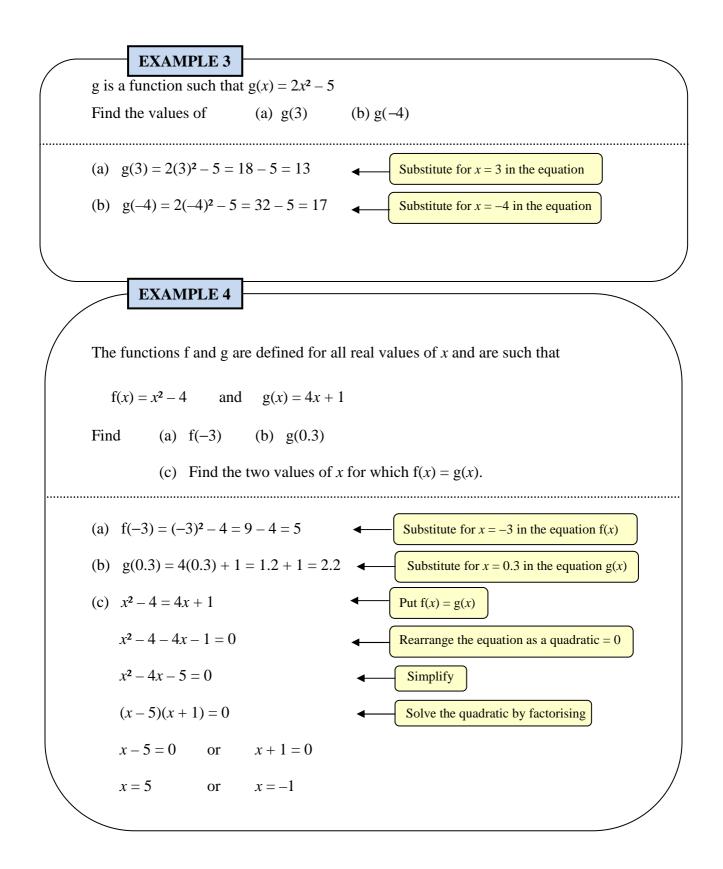
### EXAMPLE 2

f is a function such that f(x) = 3x + 4Find the values of (a) f(4) (b) f(-6)

- (a) f(4) = 3(4) + 4 = 12 + 4 = 16
- (b) f(-6) = 3(-6) + 4 = -18 + 4 = -14

Substitute for 
$$x = -6$$
 in the equation

Substitute for x = 4 in the equation



# **EXERCISE 1:**

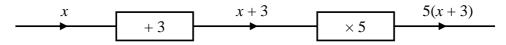
1.	The function f is such that $f(x) = 5x + 2$						
	Find	(a)	f(3)	(b)	f(7)	(c)	f(-4)
		(d)	f(-2)	(e)	f(-0.5)	(f)	f(0.3)
2.	The fu	The function f is such that $f(x) = x^2 - 4$					
	Find	(a)	f(4)	(b)	f(6)	(c)	f(-2)
		(d)	f(-6)	(e)	f(-0.2)	(f)	f(0.9)
3.	The function g is such that $g(x) = x^3 - 3x^2 - 2x + 1$						
	Find	(a)	g(0)	(b)	g(1)	(c)	g(2)
		(d)	g(-1)	(e)	g(-0.4)	(f)	g(1.5)
4.	The function f is such that $f(x) = \sqrt{2x+5}$						
	Find	(a)	f(0)	(b)	f(1)	(c)	f(2)
		(d)	f(-1)	(e)	f(-0.7)	(f)	f(1.5)
5.	f(x) =	$3x^{2}$ –	-2x - 8				
	Express $f(x + 2)$ in the form $ax^2 + bx$						
6.	The fu	The functions f and g are such that f(x) = 3x - 5 and $g(x) = 4x + 1$					
	<ul> <li>(a) Find (i) f(-1) (ii) g(2)</li> <li>(b) Find the two values of x for which f(x) = g(x).</li> </ul>						
7.	The fu	inctio	ons f and g are suc $f(x) = 2x^2 - 1$		d g(x) = 5x + 2	2	

- (a) Find f(-3) and g(-5)
- (b) Find the two values of x for which f(x) = g(x).

#### **COMPOSITE FUNCTIONS**

A composite function is a function consisting of 2 or more functions.

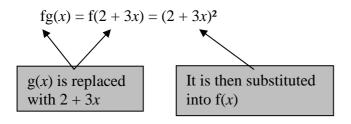
The term composition is used when one operation is performed after another operation. For instance:



This function can be written as f(x) = 5(x + 3)

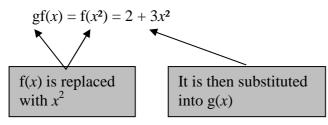
Suppose  $f(x) = x^2$ andg(x) = 2 + 3xWhat is fg(x)?Now fg(x) = f[g(x)]

This means apply g first and then apply f.



### What is gf(x)?

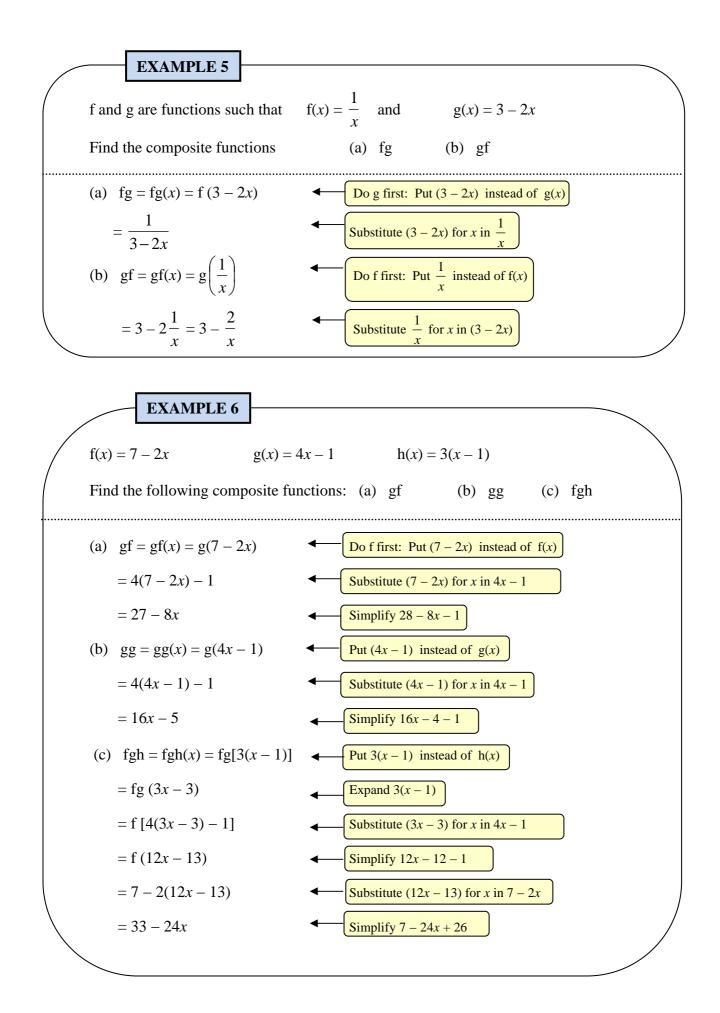
This means apply f first and then apply g.

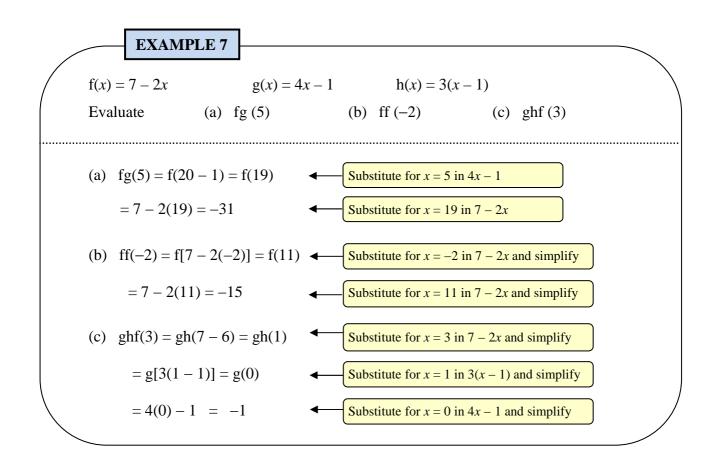


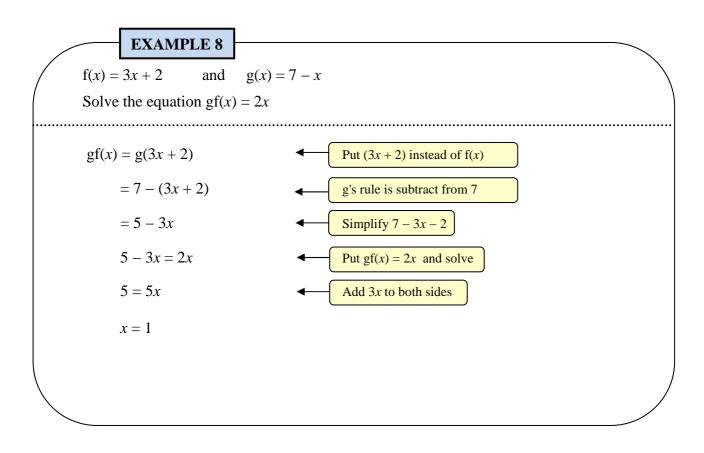
**NOTE:** The composite function gf(x) means apply f first followed by g.

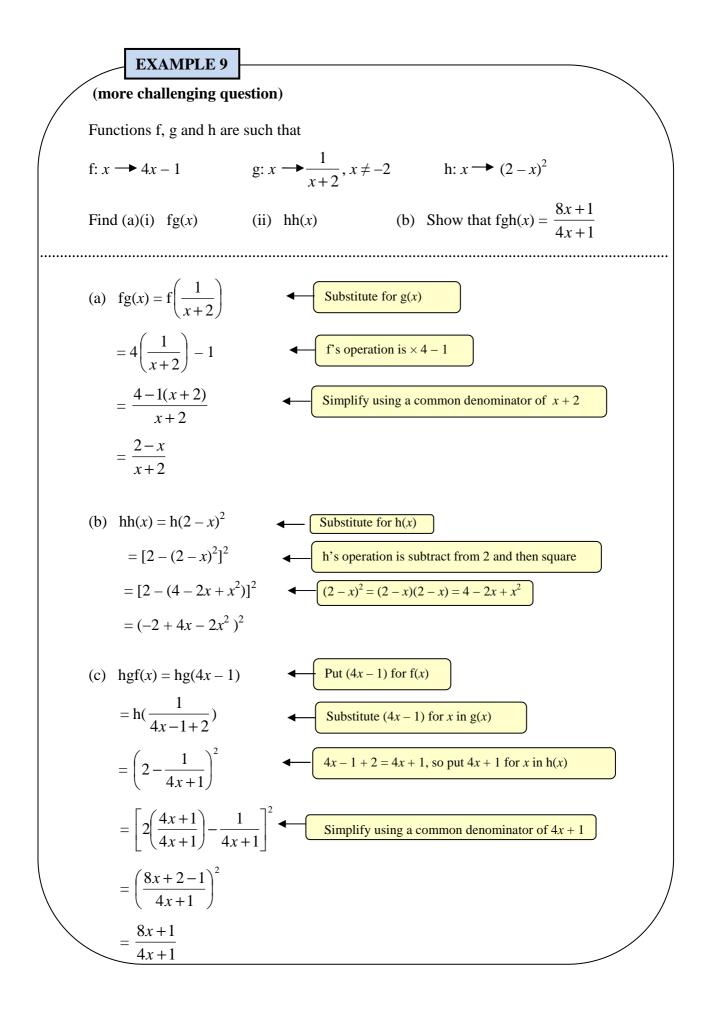
**NOTE:** The composite function fg(x) means apply g first followed by f.

**NOTE:** fg(x) can be written as fg and gf(x) can be written as gf; fg is not the same as gf.









#### **EXERCISE 2:**

- 1. Find an expression for fg(x) for each of these functions:
  - (a) f(x) = x 1 and g(x) = 5 2x(b) f(x) = 2x + 1 and g(x) = 4x + 3(c)  $f(x) = \frac{3}{x}$  and g(x) = 2x - 1(d)  $f(x) = 2x^2$  and g(x) = x + 3
- 2. Find an expression for gf(x) for each of these functions:
  - (a) f(x) = x 1 and g(x) = 5 2x(b) f(x) = 2x + 1 and g(x) = 4x + 3(c)  $f(x) = \frac{3}{x}$  and g(x) = 2x - 1(d)  $f(x) = 2x^2$  and g(x) = x + 3
- 3. The function f is such that f(x) = 2x 3Find (i) ff(2) (ii) Solve the equation ff(a) = a
- 4. Functions f and g are such that

 $f(x) = x^2$  and g(x) = 5 + xFind (a)(i) fg(x) (ii) gf(x)

(b) Show that there is a single value of x for which fg(x) = gf(x) and find this value of x.

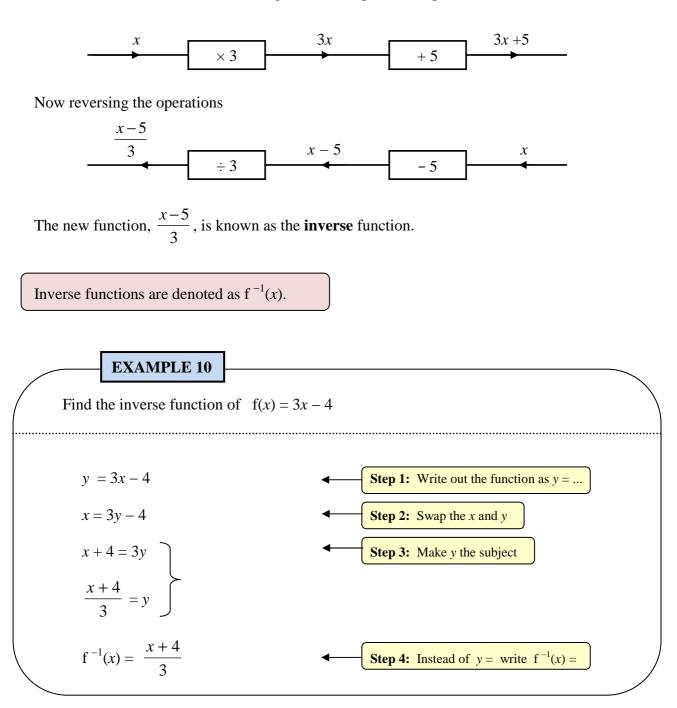
- 5. Given that f(x) = 3x 1,  $g(x) = x^2 + 4$  and fg(x) = gf(x), show that  $x^2 x 1 = 0$
- 6. The function f is defined by  $f(x) = \frac{x-1}{x}, x \neq 0$ Solve ff(x) = -2

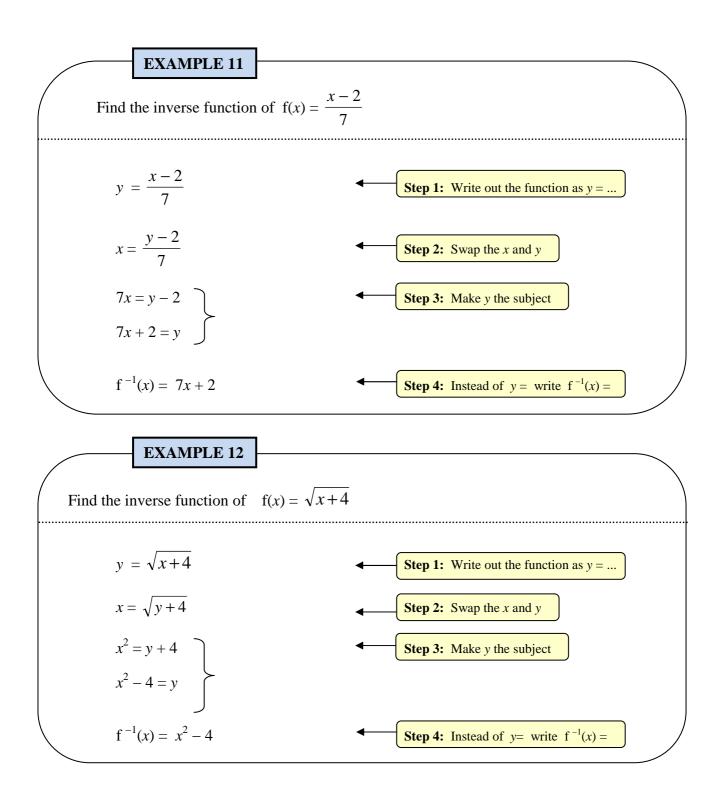
7. The function g is such that 
$$g(x) = \frac{1}{1-x}$$
 for  $x \neq 1$   
(a) Prove that  $gg(x) = \frac{x-1}{x}$   
(b) Find  $ggg(3)$ 

8. Functions f, g and h are such that f(x) = 3 - x,  $g(x) = x^2 - 14$  and h(x) = x - 2Given that f(x) = gfh(x), find the values of x.

### **INVERSE FUNCTIONS**

The function f(x) = 3x + 5 can be thought of as a sequence of operations as shown below





# **RULES FOR FINDING THE INVERSE** $f^{-1}(x)$ :

**Step 1:** Write out the function as y = ...

**Step 2:** Swap the *x* and *y* 

**Step 3:** Make *y* the subject

**Step 4:** Instead of y= write  $f^{-1}(x) =$ 

**EXERCISE 3:** 

- 1. Find the inverse function,  $f^{-1}(x)$ , of the following functions:
  - (a) f(x) = 3x 1(b) f(x) = 2x + 3(c) f(x) = 1 2x(d)  $f(x) = x^2 + 5$ (e) f(x) = 6(4x 1)(f) f(x) = 4 x(g)  $f(x) = 3x^2 2$ (h) f(x) = 2(1 x)
  - (i)  $f(x) = \frac{2}{x+1}$  (j)  $f(x) = \frac{x+1}{x-2}$
- 2. The function f is such that f(x) = 7x 3(a) Find f<sup>-1</sup>(x).
  - (b) Solve the equation  $f^{-1}(x) = f(x)$ .
- 3. The function f is such that  $f(x) = \frac{8}{x+2}$ 
  - (a) Find  $f^{-1}(x)$ .
  - (b) Solve the equation  $f^{-1}(x) = f(x)$ .
- 4. The function f is such that  $f(x) = \frac{1}{x+4}$ ,  $x \neq -4$ . Evaluate f<sup>-1</sup>(x). [Hint: First

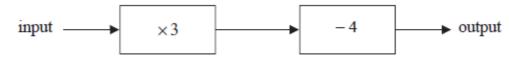
[Hint: First find  $f^{-1}(x)$  and then substitute for x = -3]

- 5.  $f(x) = \frac{x}{x+3}, x \in \mathbb{R}, x \neq -3$ (a) If  $f^{-1}(x) = -5$ , find the value of *x*.
  - (b) Show that  $\text{ff}^{-1}(x) = x$
- 6. Functions f and g are such that

f(x) = 3x + 2  $g(x) = x^2 + 1$ Find an expression for  $(fg)^{-1}(x)$  [Hint: First find fg(x)]

### **MIXED EXERCISE:**

1. Here is a number machine.



- (a) Work out the **output** when the input is 4
- (b) Work out the **input** when the output is 11
- (c) Show that there is a value of the input for which the input and the output have the same value.
- 2. Functions f and g is such that f(x) = 2x 1 and  $g(x) = \frac{3}{x}$ 
  - (a) Find the value of
    - (i) f(3)
    - (ii) fg(6)
  - (b) Express the inverse function in the form  $f^{-1}(x) = \dots$
  - (c) Express the composite function gf in the form  $gf(x) = \dots$
- 3. The function f is such that f(x) = 4x 1
  - (a) Find  $f^{-1}(x)$

The function g is such that  $g(x) = kx^2$  where k is a constant.

(b) Given that fg(2) = 12, work out the value of k

4. Functions f and g are such that f(x) = 3(x-4) and  $g(x) = \frac{x}{5} + 1$ 

- (a) Find the value of f(10)
- (b) Find  $g^{-1}(x)$
- (c) Show that ff(x) = 9x 48
- 5. Given that  $f(x) = x^2$  and g(x) = x 6, solve the equation  $fg(x) = g^{-1}(x)$
- 6. f and g are functions such that f(x) = 2x 3 and  $g(x) = 1 + \sqrt{x}$ 
  - (a) Calculate f(-4)
  - (b) Given that f(a) = 5, find the value of *a*.
  - (c) Calculate gf(6).
  - (d) Find the inverse function  $g^{-1}(x)$ .

7. Functions f and g are such that

$$f(x) = \frac{1}{x+2}$$
 and  $g(x) = \sqrt{x-1}$ 

- (a) Calculate fg(10)
- (b) Find the inverse function  $g^{-1}(x)$ .
- 8. Functions f and g are such that

f(x) = 2x + 2 and g(x) = 2x - 5

- (a) Find the composite function fg.Give your answer as simply as possible.
- (b) Find the inverse function  $f^{-1}(x)$ .
- (c) Hence, or otherwise, solve  $f^{-1}(x) = g^{-1}(x)$ .
- 9. The function f is such that  $f(x) = \frac{1}{x+3}$ 
  - (a) Find the value of f(2)
  - (b) Given that  $f(a) = \frac{1}{10}$ , find the value of *a*.

The function g is such that g(x) = x + 2

(c) Find the function gf.

Give your answer as a single algebraic fraction in its simplest form.

10. Functions f and g are such that f(x) = x<sup>2</sup> and g(x) = x - 3
(a) Find gf(x).

- (b) Find the inverse function  $g^{-1}(x)$ .
- (c) Solve the equation  $gf(x) = g^{-1}(x)$ .
- 11. The function f is such that  $f(x) = (x 1)^2$ 
  - (a) Find f(8)

The function g is such that  $g(x) = \frac{x}{x-1}$ 

- (b) Solve the equation g(x) = 1.2
- (c) (i) Express the inverse function  $g^{-1}$  in the form  $g^{-1}(x) = \dots$ 
  - (ii) Hence write down gg(x) in terms of x.

- 12. f is a function such that  $f(x) = \frac{1}{x^2 + 1}$ 
  - (a) Find  $f(\frac{1}{2})$

g is a function such that  $g(x) = \sqrt{x-1}$ ,  $x \ge 1$ 

- (b) Find fg(x)Give your answer as simply as possible.
- 13. The function f is such that  $f(x) = \frac{x-6}{2}$ 
  - (a) Find f(8)
  - (b) Express the inverse function  $f^{-1}$  in the form  $f^{-1}(x) = \dots$

The function g is such that  $g(x) = \sqrt{x-4}$ 

- (c) Express the function gf in the form gf(x) = ...Give your answer as simply as possible.
- 14. Functions f and g are such that f(x) = 3x 2 and  $g(x) = \frac{10}{x+2}$ 
  - (a) Express the inverse function  $f^{-1}$  in the form  $f^{-1}(x) = ...$
  - (b) Find gf(x)Simplify your answer.
- 15. Functions f and g are such that  $f(x) = \frac{2}{x}$  and  $g(x) = \frac{x+1}{x}$ 
  - (a) Solve gf(a) = 3
  - (b) Express the inverse function  $g^{-1}$  in the form  $g^{-1}(x) = ...$

16. Functions g and h are such that  $g(x) = \frac{x}{2x-5}$  and h(x) = x+4

- (a) Find the value of g(1)
- (b) Find gh(x)Simplify your answer.
- (c) Express the inverse function  $g^{-1}$  in the form  $g^{-1}(x) = \dots$

## ANSWERS

### Exercise 1

1.	(a) 17	(b) 37	(c) -18
	(d) -8	(e) -0.5	(f) 3.5
2.	(a) 12	(b) 32	(c) 0
	(d) 32	(e) -3.96	(f) -3.19
3.	(a) 1	(b) -3	(c) -7
	(d) -1	(e) 1.256	(f) -5.375
4.	(a) $\sqrt{5}$	(b) $\sqrt{3}$	(c) 3
	(d) $\sqrt{3}$	(e) $\frac{3\sqrt{10}}{5}$	(f) $2\sqrt{2}$

5.  $3x^2 + 10x$ 

6. (a) 
$$f(-1) = 8$$
 and  $g(2) = 9$   
7. (a)  $f(-3) = 17$  and  $g(-5) = -23$   
(b)  $x = -\frac{1}{2}$  and  $x = 3$ 

## Exercise 2

1.	(a) $4 - 2x$	(b) $8x + 17$	(c) $\frac{3}{2x-1}$	(d) $2(x+3)^2$	
2.	(a) $7 - 2x$	(b) $8x + 7$	(c) $\frac{6}{x} - 1$	(d) $2x^2 + 3$	
3.	(a) – 1	(b) <i>a</i> = 3			
4.	(a)(i) $(5+x)^2$	(ii) $5 + x^2$	(b) $x = -2$		
5.	$3(x^{2} + 4) - 1 = (3x - 1)^{2} + 4$ $3x^{2} + 12 - 1 = 9x^{2} - 6x + 1 + 4$ $3x^{2} + 11 = 9x^{2} - 6x + 5$ $6x^{2} - 6x - 6 = 0$ $x^{2} - x - 1 = 0$				
6.	$x = \frac{3}{2}$				
7.	(a) $g\left(\frac{1}{1-x}\right) = \frac{1}{1-\frac{1}{1-x}}$	$\frac{1}{x} = \frac{1}{\frac{1-x-1}{1-x}} = \frac{1-x}{-x} = \frac{x-1}{x}$	(b) $-\frac{1}{4}$		

8. x = 1 and x = 8

### **Exercise 3**

1. (a) 
$$\frac{x+1}{3}$$
 (b)  $\frac{x-3}{2}$  (c)  $\frac{1-x}{2}$  (d)  $\pm\sqrt{x-5}$   
(e)  $\frac{x+6}{24}$  (f)  $4-x$  (g)  $\pm\sqrt{\frac{x+2}{3}}$  (h)  $\frac{2-x}{2}$   
(i)  $\frac{2-x}{x}$  (j)  $\frac{2x+1}{x-1}$   
2. (a)  $\frac{x+3}{7}$  (b)  $x=0.5$   
3. (a)  $\frac{8}{x}-2$  (b)  $x=-4$  and  $x=2$   
4.  $-\frac{13}{3}$   
5. (a)  $\frac{5}{2}$  (b)  $f\left(\frac{3x}{1-x}\right) = \frac{\frac{3x}{1-x}}{\frac{3x}{1-x}+3} = \frac{\frac{3x}{1-x}}{\frac{3x+3(1-x)}{1-x}} = \frac{3x}{3} = x$   
6.  $\pm\sqrt{\frac{x-5}{3}}$ 

### **Mixed Exercise**

1. (a) 8 (b) 29 (c) 2 2. (a) (i) 5 (ii) 0 (b)  $\frac{x+1}{2}$  (c)  $\frac{3}{x}$ 3. (a)  $\frac{x+1}{4}$  (b)  $k = \frac{13}{16}$ 4. (a) 18 (b) 5(x-1)(c) ff(x) = 3(3(x-4)-4) = 3(3x-12-4) = 3(3x-16) = 9x - 485. x = 6 and x = 76. (a) -11 (b) a = 4 (c) 4 (d)  $(x-1)^2$ 

7.	(a) $\frac{1}{5}$	(b) $x^2 + 2$	
8.	(a) $6x - 13$	(b) $\frac{x-2}{3}$	(c) $x = -19$
9.	(a) $\frac{1}{5}$	(b) <i>a</i> = 7	(c) $\frac{2x+7}{x+3}$
10.	(a) $x^2 - 3$	(b) $x + 3$	(c) $x = -2$ and
11.	(a) 49	(b) $x = 6$	(c) (i) $\frac{x}{x-1}$
12.	(a) 0.8	(b) $\frac{1}{x}$	
13.	(a) 1	(b) $2x + 6$	(c) $\sqrt{\frac{x-14}{2}}$
14.	(a) $\frac{x+2}{3}$	(b) $\frac{10}{3x}$	
15.	(a) $x = 4$	(b) $\frac{1}{x-1}$	
16.	(a) $-\frac{1}{3}$	(b) $\frac{x+4}{2x+3}$	(c) $\frac{5x}{2x-1}$

= -2 and x = 3 $\frac{x}{x-1}$ (ii) x