MEI

A Level

A Level Mathematics

C2 Sequences and Series

(Answers)

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Total Marks: /73

C2 - Sequences and Series (Answers)
MEI

1. Consider an arithmetic sequence with k^{th} term given by $a_k = a + (k-1)d$. Prove that

$$S = \sum_{k=1}^{n} a_k = \frac{1}{2}n(2a + (n-1)d).$$

To begin, write out the sum term by term:

$$S = a + (a+d) + (a+2d) + (a+3d) + \cdots + (a+(n-2)d + (a+(n-1)d).$$

Now compute $S+S$ using the above by adding the first term to the last term, the second term to the second-to-last term, the third term to the third-to-last term and so on. To make this easier to visualise we may write:

$$S = a + (a+d) + \cdots + (a+(n-2)d + (a+(n-1)d). \quad (1)$$
$$S = (a+(n-1)d) + (a+(n-2)d) + \cdots + (a+d) + a. \quad (2)$$

We now add both lines together to obtain:

$$2S = (2a+(n-1)d) + (2a+(n-1)d) + \cdots + (2a+(n-1)d) + (2a+(n-1)d),$$

Where we note that we simply have n lots of $(2a+(n-1)d)$ on the right. And so we write:

$$2S = n(2a+(n-1)d),$$

which when rearranged yields:

$$S = \frac{1}{2}n(2a+(n-1)d),$$

as required. [8]

2. Consider the sequence defined recursively by:

$$u_{n+2} = 4u_{n+1} - u_n, \qquad n \geq 1,$$

where,

$$u_1 = 1, \quad u_2 = 2.$$

(a) $u_3 = 7$ and $u_4 = 26$. [3]

(b) $\sum_{n=1}^{5} u_n = 1 + 2 + 7 + 26 + 97 = 133$. [3]

3. Consider the sequence defined recursively by:

$$u_{n+2} = 2u_{n+1} - u_n, \qquad n \geq 1,$$

where,

$$u_1 = 5, \quad u_2 = 7.$$

(a) $u_3 = 9, u_4 = 11$ and $u_5 = 13$. [3]

(b) $\sum_{n=1}^{5} u_n = 5 + 7 + 9 + 11 + 13 = 45$. [2]

(c) We find the n^{th} term of the sequence to be $u_n = 2n + 3$. [4]

(d) This is a hard question. To calculate this sum we adopt the method used in question 1). Let $S = \sum_{n=1}^{100} u_n$. We write this out term by term as:

$$S = 5 + 7 + 9 + 11 + \cdots + 201 + 203. \qquad (3)$$

Now we rewrite S again but this time order the terms from largest to smallest rather than smallest to largest:

$$S = 203 + 201 + 199 + 197 + \cdots + 7 + 5. \qquad (4)$$

We now add these two lines together term by term (in other words we do (3)+(4)) to get:

$$2S = 208 + 208 + 208 + 208 + \cdots + 208 + 208.$$

Now there are 100 terms on the right hand side and so we have:

$$2S = 100 \times 208,$$

which we rearrange to get:

$$S = 50 \times 208 = 10400.$$

Therefore we get the answer:

$$S = \sum_{n=1}^{100} u_n = 10400.$$

[5]

4. The k^{th} term of an arithmetic sequence is given by $a_k = 2 + 4(k-1)$.

   (a) $a_1 = 2$. [1]

   (b) $a_2 = 6$. [2]

   (c) $d = 4$. [1]

   (d) $\sum_{k=1}^{2} a_k = a_1 + a_2 = 2 + 6 = 8$. [1]

   (e) $\sum_{k=1}^{50} a_k = 5000$ *(Use the arithmetic series formula given in Q1 with $n = 50$, $a = 2$ and $d = 4$)* [3]

5. An arithmetic sequence has fifth term $a_5 = 17$ and eigth term $a_8 = 26$.

   (a) $d = 3$ and $a = 5$. [2]

   (b) $\sum_{k=1}^{30} a_k = 1455$. [3]

6. Consider the sequence with n^{th} term given by $u_n = 2n + 1$.

   (a) $u_1 = 3, u_2 = 5, u_3 = 7$ and $u_4 = 9$. [2]

   (b) This is an arithmetic sequence with common difference $d = 2$ and first term $a = 3$. [2]

   (c) $\sum_{n=1}^{20} u_n = 440$. [3]

7. Which of the following series are convergent? Which are divergent? *You are not required to evaluate the convergent series.*

   (a) $\sum_{n=0}^{\infty} 1$ is divergent. We are adding the number 1 to itself an infinite number of times. [1]

   (b) $\sum_{n=0}^{\infty} 2n$ is divergent. The numbers are getting bigger and bigger each time and we are adding an infinite number of them. [1]

   (c) $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ is convergent. This is a geometric series with $r = \frac{1}{2}$. Since $\left|\frac{1}{2}\right| < 1$, it converges. [1]

   (d) $\sum_{n=0}^{\infty} 20 \left(\frac{1}{2}\right)^n$ is convergent. This is just a scalar multiple of the series above. [1]

   (e) $\sum_{n=0}^{\infty} 2^n$ is divergent. This is a geometric series with $r = 2$. Since $|2| > 1$, it diverges. [1]

   (f) $\sum_{n=0}^{\infty} r^n$ is a geometric series. It converges if $|r| < 1$. [2]

   (g) $\sum_{n=0}^{\infty} (2r)^n$ is a geometric series. It converges if $|2r| < 1$. I.e. when $|r| < \frac{1}{2}$. [2]

8. Consider the geometric sequence with general term given by $a_k = \left(\frac{1}{2}\right)^k$:

   (a) $a_3 = \frac{1}{8}$. [1]

   (b) $\sum_{k=0}^{20} a_k = \frac{1-\left(\frac{1}{2}\right)^{20}}{1-\frac{1}{2}} = \frac{2097151}{1048576} = 1.999\cdots$. [3]

   (c) $\sum_{k=0}^{\infty} a_k = 2$. [3]

9. Be careful! This is a sum from 1 to ∞ not 0 to ∞. To begin, we evaluate $\sum_{k=0}^{\infty} 4\left(\frac{1}{6}\right)^k = \frac{24}{5}$ as normal. But we need to spot that we have added one term too many. We need to discount the zero term. The zero term is $4 \times \left(\frac{1}{6}\right)^0 = 4$. And so we get the answer:

$$\sum_{k=1}^{\infty} 4\left(\frac{1}{6}\right)^k = \frac{24}{5} - 4 = \frac{4}{5}$$

[4]

10. Consider a geometric sequence with k^{th} term $a_k = ar^k$ such that:

$$a_1 = 1,$$
$$\sum_{k=0}^{\infty} ar^k = \frac{9}{2}.$$

   (a) Either $a = 3$ and $r = \frac{1}{3}$, or $a = \frac{3}{2}$ and $r = \frac{2}{3}$. [5]