AQA, Edexcel, OCR

A Level

A Level Mathematics

Newton-Raphson method and other recurrence relations (Answers)

Name:

MME
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Total Marks:

C2- Newton-Raphson method and other recurrence relations- Answers

AQA, Edexcel, OCR

1) **Write the first four terms of the recurrence relationship defined as**

$$\boldsymbol{U_{n+1} = 3U_n + 1}$$

where $\boldsymbol{U_0 = 3}$

[1 mark for correct initialisation]

$$U_1 = 3(3) + 1 = 10$$

$$U_2 = 3(10) + 1 = 91$$

$$U_3 = 3(91) + 1 = 274$$

[1 mark for correct answer]

$$U_4 = 3(274) + 1 = 823$$

2) **A relationship is given as**

$$\boldsymbol{R_{n+1} = (AR_n + B)}$$

we know that

$\boldsymbol{R_0 = 4, R_1 = 6, R_2 = 8, R_3 = 10}$

Determine a general solution and the value for $\boldsymbol{R_4}$.

[1 mark]

Substituting the known values of R_n gives the statements **(1)**

$$6 = A(4) + B \quad \textbf{(2)}$$

$$8 = A(6) + B \quad \textbf{(3)}$$

$$10 = A(8) + B$$

Subtracting (1) from (2) gives

$$2 = 2A \quad \textbf{(4)}$$

$$\Rightarrow A = 1$$

Substituting (4) into (3) gives

$$10 = 8 + B \quad \textbf{(5)}$$

$$\Rightarrow B = 2$$

[1 mark for correct answer]

Equations (4) and (5) allow us to create a general solution of

$$R_{n+1} = R_n + 2$$

$$\Rightarrow R_4 = 10 + 2$$

$$R_4 = 12$$

3) **i) Use the Newton-Raphson method to find the first four terms of the following:**

$$\boldsymbol{x^3 + 3x^2 - 8x + 0.8 = 0}$$

You may use

$$\boldsymbol{x_0 = 0}$$

Newton-Raphson method is defined as

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$$

[1 mark]

Here,

$$f(x) = x^3 + 3x^2 - 8x + 0.8$$

$$f'(x) = 3x^2 + 6x - 8$$

[1 mark]

Initialise the iteration process with $x_0 = 0$

$$x_1 = 0 - \frac{0^3 + 3(0)^2 - 8(0) + 0.8}{0 + 0 - 8} = \frac{0.8}{-8} = -0.1$$

$$x_2 = -0.1 - \frac{-0.1^3 + 3(-0.1)^2 - 8(-0.1) + 0.8}{3(-0.1)^2 + 6(-0.1) - 8} = 0.104206$$

[1 mark]

The solution continued using a spreadsheet

n	x(n)	f(x)	f'(x)	x(n+1)
2	0.104206242	5.84596E-05	-7.342185729	0.104214204
3	0.104214204	2.10007E-10	-7.342132977	0.104214204
4	0.104214204	0	-7.342132977	**0.104214204**

ii) Explain why $\boldsymbol{x_0 = \sqrt{\frac{11}{3}} - 1}$ is not a viable option.

[1 mark]

Substituting this value in gives

$$x_1 = \sqrt{\frac{11}{3}} - 1 \;\; - \frac{\left(\sqrt{\frac{11}{3}} - 1\right)^3 + 3\left(\sqrt{\frac{11}{3}} - 1\right)^2 - 8\left(\sqrt{\frac{11}{3}} - 1\right) + 0.8}{3\left(\sqrt{\frac{11}{3}} - 1\right)^2 + 6\left(\sqrt{\frac{11}{3}} - 1\right) - 8}$$

[1 mark]

If we concentrate on the expanding the denominator only, we get

$$3\left(\frac{11}{3} - 2\sqrt{\frac{11}{3}} + 1\right) + 6\sqrt{\frac{11}{3}} - 6 - 8$$

$$= 11 - 6\sqrt{\frac{11}{3}} + 3 + 6\sqrt{\frac{11}{3}} - 14$$

$$= 14 - 14 = 0$$

And we cannot have a 0 denominator. Therefore, it will not work.

4) What is the value of the term U_{56} for the relationship

$$\boldsymbol{U_{n+1} = -Un^{n}}$$

where $U_0 = 1$

The first four terms are as allows

[1 mark]

$$U_1 = -1^0 = -1$$

$$U_2 = -(-1)^1 = 1$$

$$U_3 = -1^2 = -1$$

$$U_4 = -(-1)^3 = 1$$

[1 mark]

The pattern is for U_{2n}= 1 and $U_{2n+1} = -1$. Therefore,

$$U_{56} = 1$$

5) **i) Draw a flow chart showing how to estimate a solution to $x^2 + 3x - 6 = 0$ using a recurrence process.**

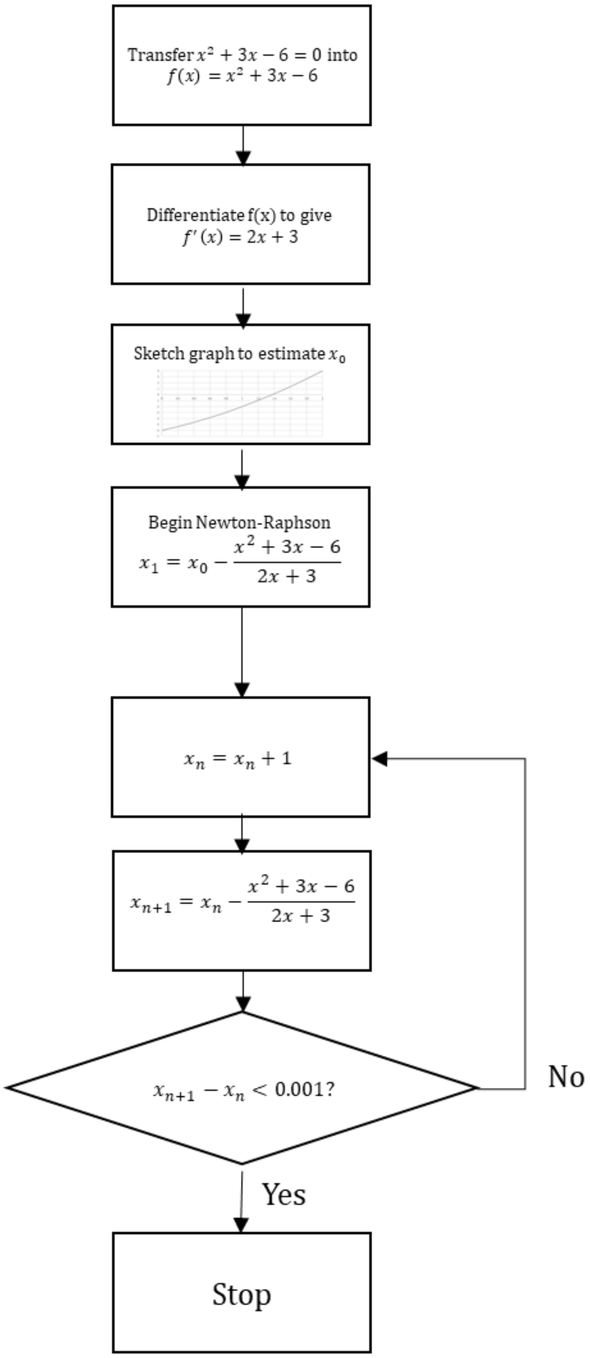
[1 mark for correct differential step]

[1 mark for estimating a solution graphically or other]

[1 mark for correct initialisation of Newton-Raphson or other]

[1 mark for continuous loop until answer stops changing]

Some form of the following graph including the steps of initialisation and recurrence until a desired level of decimal points in an answer are obtained.



ii) Calculate one of the roots to 2dp.

[1 mark]

Use the quadratic formula and *a=1, b=3 and c=-6*

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(-6)}}{2(1)}$$

$$= \frac{-3 \pm \sqrt{33}}{2}$$

$$x = \frac{-3 + \sqrt{33}}{2} \approx 1.37 \ (to\ 2dp)$$

[1 mark]

We only need one of the solutions, however, an alternative could be

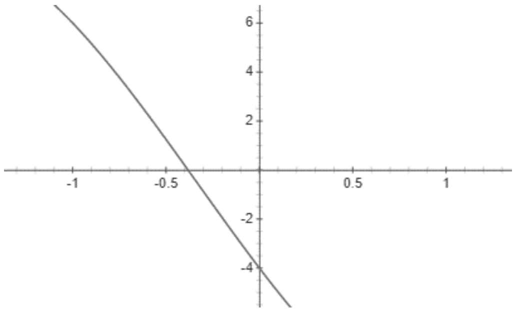
$$x = \frac{-3 - \sqrt{33}}{2} \approx -4.37 \ (to\ 2dp)$$

6) **i) Using an iterative process find one of the non-integer roots of**

$$\mathbf{2x^3 + 2x^2 - 10x = 4} \qquad \textbf{(1)}$$

[1 mark]

Using the Newton-Raphson method, we need to obtain an initial guess for x_n. Sketching the graph gives us



Attempting $x_n = -0.4$, seems sensible.

[1 mark]

The function (1) can be written as

$$f(x) = 2x^3 + 2x^2 - 10x - 4$$

$$\therefore f'(x) = 6x^2 + 4x - 10$$

[1 mark]

Using Newton-Raphson gives

$$x_{n+1} = x_n - \frac{2x^3 + 2x^2 - 10x - 4}{6x^2 + 4x - 10}$$

$$x_1 = -0.4 - \frac{2(-0.4)^3 + 2(-0.4)^2 - 10(-0.4) - 4}{6(-0.4)^2 + 4(-0.4) - 10} = -0.3819$$

[1 mark]

The next few iterations show that a solution is found readily

n	x(n)	f(x)	f'(x)	x(n+1)
0	-0.4	0.192	-10.64	-0.381954887
1	-0.381954887	-0.000118499	-10.65248233	-0.381966011
2	-0.381966011	-3.61027E-11	-10.65247584	-0.381966011
3	-0.381966011	0	-10.65247584	-0.381966011
4	-0.381966011	0	-10.65247584	**-0.381966011**

ii) Show that one of the roots is 2.

$$2x^3 + 2x^2 - 10x = 4$$

[1 mark]

Substitute in $x = 2$ into (1)

$$2(2)^3 + 2(2)^2 - 10(2) = 4$$

$$16 + 8 - 20 = 4$$

$$24 - 20 = 4$$

7) Use the Newton-Raphson method to find one of the solutions to

$$\boldsymbol{x^2 + 5x - 11 = 0}$$

You may use

$$\boldsymbol{x_0 = 1.6}$$

[1 mark]

$$X_1 = 1.6 - \frac{(1.6)^2 + 5(1.6) - 11}{2(1.6) + 5} = 1.653$$

[1 mark]

n	x(n)	f(x)	f'(x)	x(n+1)
0	1.6	-0.44	8.2	1.653658537
1	1.653658537	0.002879239	8.307317073	1.653311946
2	1.653311946	1.20125E-07	8.306623892	1.653311931
3	1.653311931	0	8.306623863	1.653311931
4	1.653311931	0	8.306623863	**1.653311931**

8) **Estimate $\sqrt{2}$ using the Newton Raphson method.**

[1 mark]

To estimate this, we need to define a function to which $\sqrt{2}$ is a solution. This function is:

$$x^2 - 2 = 0$$

[1 mark]

Continuing with Newton-Raphson as normal

$$f(x) = x^2 - 2$$

$$f'(x) = 2x$$

[1 mark]

Estimating the initial value (by sketch or recalling it)

$$x_0 = 1.4$$

$$x_1 = 1.4 - \frac{(1.4)^2 - 2}{2(1.4)} = 1.14128$$

[1 mark]

Looking at the full table of iterations we see that a more precise and consistent value is found after only 3 iterations.

n	x(n)	f(x)	f'(x)	x(n+1)
0	1.4	-0.04	2.8	1.414285714
1	1.414285714	0.000204082	2.828571429	1.414213564
2	1.414213564	5.20563E-09	2.828427128	1.414213562
3	1.414213562	0	2.828427125	1.414213562
4	1.414213562	0	2.828427125	1.414213562
5	1.414213562	0	2.828427125	1.414213562