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AQA, Edexcel, OCR

A Level

A Level Mathematics

Understand and use the standard small angle approximations of sine, cosine and tangent (Answers)

Name:

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Total Marks:

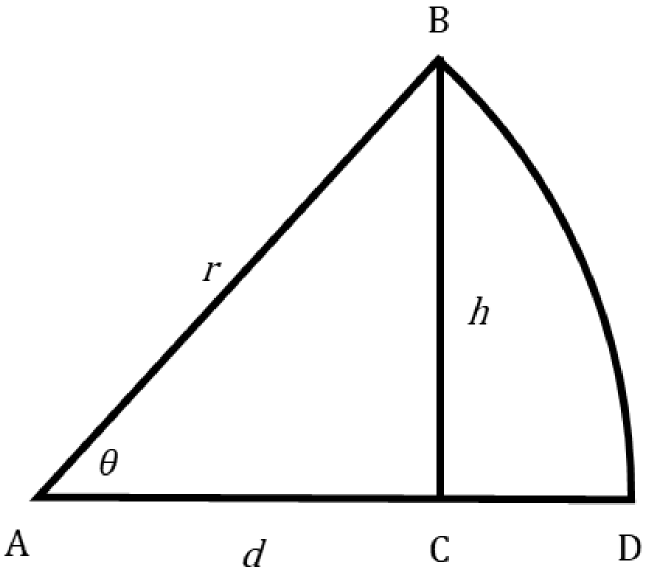
E2- Understand and use the standard small angle approximations of sine, cosine and tangent - Answers

AQA, Edexcel, OCR

1) **Sketch and derivate from it the geometric proof for the small angle approximations of sine, cosine and tangent.**

[1 mark]

Begin by sketching a circle with triangle contained within.



[1 mark]

We can obtain the length of h by

$$h = rsin\theta$$

or

$$h = rtan\theta$$

as $\theta \to 0$

$$length\ (r) \to length(d)$$

$$length(h) \to length(arc\ CD)$$

The length of CD can be calculated using

$$CD = r\theta$$

provided the *measurement is in radians.* (1) (2)

[1 mark]

Therefore, we can write

$$h = rsin\theta \approx r\theta \approx rtan\theta$$

$$\Rightarrow sin\theta \approx \theta$$

$$\Rightarrow tan\theta \approx \theta$$

[1 mark]

To obtain an estimate for $cos\theta$ use following the double angle formula

$$\cos(2x) = 1 - 2\sin^2(x)$$

where $x = \frac{\theta}{2}$ and we use the estimate for sine previously given in (1). (3)

$$\cos(\theta) = 1 - 2\sin^2\left(\frac{\theta}{2}\right)$$

$$\cos(\theta) \approx 1 - 2\left(\frac{\theta}{2}\right)^2$$

$$\Rightarrow \cos(\theta) \approx 1 - \frac{\theta^2}{2}$$

2) **Give the small angle approximations for sine, cosine and tangent of:**

i) 5^{o}

ii) 10^{o}

Firstly, convert to radians. Small angle approximations only work with radians. Then use the rules as you remember them or copy them from the previous question.

[1 mark for each correct answer. 6 marks in total]

Degrees	Radians	Sine	Cosine	Tangent
5.0	0.0873	0.0873	0.9962	0.0873
10.0	0.1745	0.1745	0.9848	0.1745

3) **i) Generate a table of the small angle approximations for sine, cosine and tangent of:**]

$$\mathbf{0, \frac{\pi}{12}, \frac{\pi}{10}, \frac{\pi}{8}, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi}$$

ii) Then add an additional column and complete the actual values.

iii) Plot the actual values against the approximations on a four quadrant axes ranging from -5 to 5 for Approximation (x-axis) and Actual (y-axis).

iv) Calculate the mean absolute percentage error for sine, cosine and tangent.

[1 mark for each correctly completed table for approximations- 3 max]

[1 mark for each correctly completed table for actual values- 3 max]

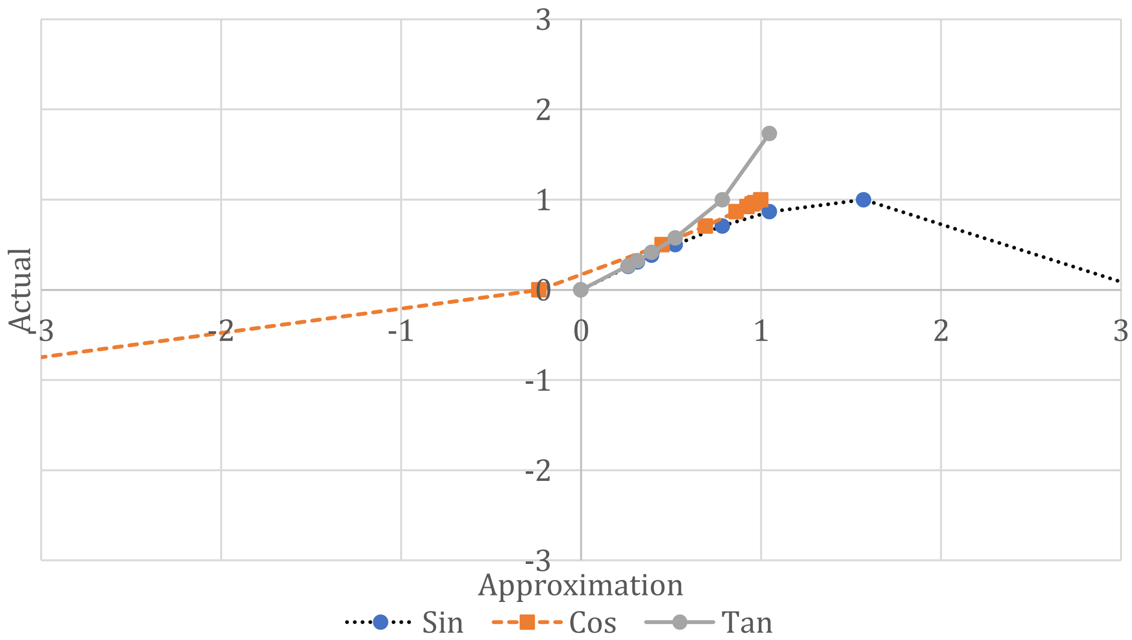
[1 mark for each correctly completed table for % error- 3 max]

		Sine		
Integer	**Precise**	**Approx.**	**Actual**	**% Error**
0	0	0	0	0%
12	0.261799388	0.261799388	0.258819045	0%
10	0.314159265	0.314159265	0.309016994	1%
8	0.392699082	0.392699082	0.382683432	1%
6	0.523598776	0.523598776	0.5	2%
4	0.785398163	0.785398163	0.707106781	8%
3	1.047197551	1.047197551	0.866025404	18%
2	1.570796327	1.570796327	1	57%
1	3.141592654	3.141592654	1.22515E-16	314%
			MAPE	**45%**

		Cosine		
Integer	**Precise**	**Approx.**	**Actual**	**% Error**
0	0	1	1	0%
12	0.261799388	0.96573054	0.965925826	0%
10	0.314159265	0.950651978	0.951056516	0%
8	0.392699082	0.922893716	0.923879533	0%
6	0.523598776	0.862922161	0.866025404	0%
4	0.785398163	0.691574862	0.707106781	2%
3	1.047197551	0.451688644	0.5	5%
2	1.570796327	-0.23370055	6.12574E-17	23%
1	3.141592654	-3.934802201	-1	293%
			MAPE	**36%**

		Tangent		
Integer	**Precise**	**Approx.**	**Actual**	**% Error**
0	0	0	0	0%
12	0.261799388	0.261799388	0.267949192	1%
10	0.314159265	0.314159265	0.324919696	1%
8	0.392699082	0.392699082	0.414213562	2%
6	0.523598776	0.523598776	0.577350269	5%
4	0.785398163	0.785398163	1	21%
3	1.047197551	1.047197551	1.732050808	68%
2	1.570796327	1.570796327		
1	3.141592654	3.141592654	-1.22515E-16	314%
			MAPE	**52%**

[1 mark for each graph drawn correctly – 3 max]
[1 mark for correct axes]



4) A function machine takes two small angle approximations and multiplies them together. Jack puts in $\sin(9^o)$ and $\cos(9^o)$. Jill puts in $\sin(8^o)$ and $\tan(11^o)$. Show who ends up with the largest answer. Do not use a calculator. You may work using two decimal places.

Firstly, convert to radians. Small angle approximations only work with radians. Then use the rules as you remember them or copy them from the previous question.

The rules are

$$sin\theta \approx \theta$$
$$tan\theta \approx \theta$$
$$\cos(\theta) \approx 1 - \frac{\theta^2}{2}$$

[1 mark for each row correctly completed- 3 max]

The values needed

Degrees	Radians	Sin	Cos	Tan
8	0.13962634	0.14		
9	0.157079633	0.16	0.9872	
11	0.191986218			0.19

[1 mark for correct answer]

Jack's answer is

$$0.16 \times 0.9872 = 0.16$$

Jill's answer is

$$0.14 \times 0.19 = 0.03$$

Jack's answer is largest.

5) Approximate the value of $A = \frac{\pi}{12}$ with the formulas:

i) $\cos(2A)$

[1 mark]

$$\cos(\theta) = 1 - 2\sin^2\left(\frac{\theta}{2}\right)$$
$$\cos(\theta) \approx 1 - 2\left(\frac{\theta}{2}\right)^2$$
$$\Rightarrow \cos(\theta) \approx 1 - \frac{\theta^2}{2}$$

$$cos2A = 1 - 2\sin^2 A$$

[1 mark]

$$\cos\left(\frac{\pi}{6}\right) \approx 2\sin^2\left(\frac{\pi}{12}\right)$$
$$\approx 2\left(\frac{\pi}{12}\right)^2$$
$$\approx \frac{\pi^2}{144}$$

ii) $\sin(2A)$

[1 mark]

$$sin(2A) = 2sin(A)cos(A)$$

[1 mark]

$$sin2A \approx \frac{\pi}{6}(1 - \frac{\frac{\pi}{12}^2}{2})$$

$$\approx \frac{\pi}{6} - \frac{\pi}{6} \cdot \frac{\pi^2}{288}$$

$$\approx \frac{\pi}{6} - \frac{\pi^3}{288}$$

$$\approx \frac{\pi}{6}(1 - \frac{\pi^2}{68})$$

iii) tan(2A)

[1 mark]

$$tan(2A) = \frac{2tan(A)}{1 - tan^2 A}$$

[1 mark]

$$tan\left(\frac{\pi}{6}\right) \approx \frac{\frac{\pi}{6}}{1 - \frac{\pi^2}{144}}$$

$$\approx \frac{\pi}{6 - \frac{\pi^2}{24}}$$

iv) sin(A)cos(A)tan(A)

[1 mark]

$$sin(A)cos(A)tan(A) \approx \left(\frac{\pi}{12}\right)\left(1 - \frac{\frac{\pi}{12}^2}{2}\right)\left(\frac{\pi}{12}\right)$$

[1 mark]

$$\approx (\frac{\pi^2}{144})(1 - \frac{\pi^2}{288})$$

6) **Your manager wants to save time but be accurate. You are allowed a 2% error in your approximations otherwise you must find the precise value. For $\sin(x)$:**

i) **What integer angles, in degrees, would you not be allowed to approximate? Write your answer as an inequality.**

This requires a little trial and improvement.

And results in the answer

$$x > 13^o$$

The derivation of that answer is shown in the table below.

[1 mark to establish between 13 and 14]

[1 mark for correct inequality]

Degrees	Radians (and estimate)	Actual $\sin(x)$	$\frac{\lvert Estimate - Actual\rvert}{Actual} \times 100$
13	0.340339204	0.333806859	1.95692351
14	0.366519143	0.35836795	2.274531912
13.5	0.353429174	0.346117057	2.112613725

ii) **You are required to work out all the integer values of $\sin(x)$ from 1^o to 100^o Approximations take you 5 seconds, calculations take you 15 seconds, how long will this task take in total?**

[1 mark]

$$Time = 5 \times 13 + 15 \times 87$$
$$= 65 + 1305$$
$$= 1370\ seconds$$
$$= 22m30s$$

iii) **If you were offered the swap to $\cos(x)$ or $\tan(x)$, would you? And why?**

[1 mark each for statement about tan and cos- 2 max]

Tan is the easiest to calculate first as the estimates are the same as Tan. In this instance only the first 9 degrees are within a 2% error, meaning a longer time to work them out. Similarly cos also takes longer as only the first 9 degrees are within the 2% error, again, meaning it would take longer to calculate them than sin.