A24 RECOGNISE AND USE SEQUENCES OF TRIANGULAR, SQUARE AND CUBE NUMBERS, SIMPLE ARITHMETIC PROGRESSIONS, FIBONACCI TYPE SEQUENCES, QUADRATIC SEQUENCES, AND SIMPLE GEOMETRIC PROGRESSIONS (r^n WHERE n IS AN INTEGER, AND r IS A RATIONAL NUMBER > 0 **OR A SURD**) **AND OTHER SEQUENCES (higher tier)**

SPECIAL SEQUENCES

Before we look at higher level sequences, here are some important sequences that you should know.

Odd numbers

1 3 5 7 9 11 .... $2n - 1$

Even numbers

2 4 6 8 10 12 .... $2n$

Square numbers

1 4 9 16 25 36 .... n^2

Cube numbers

1 8 27 64 125 216 .... n^3

Triangle numbers

1 3 6 10 15 21 .... $\frac{1}{2}n(n + 1)$

Prime numbers

2 3 5 7 11 13 .... ?

Note: there is no formula for calculating the *n*th term for prime numbers.

Fibonacci

1 1 2 3 5 8 13 21 ....

The Fibonacci sequence

The Fibonacci numbers are nature's numbering system. They appear everywhere in nature, from the leaf arrangement in plants, to the pattern of the florets of a flower, the bracts of a pinecone, or the scales of a pineapple. The Fibonacci numbers are therefore applicable to the growth of every living thing, including a single cell, a grain of wheat, a hive of bees, and even all of mankind.

Many plants show the Fibonacci numbers in the arrangement of the leaves around the stem. Some pine cones and fir cones also show the numbers, as do daisies and sunflowers. Sunflowers can contain the number 89, or even 144.

Many other plants, such as succulents, also show the numbers. Some coniferous trees show these numbers in the bumps on their trunks and palm trees show the numbers in the rings on their trunks

The Fibonacci numbers are the numbers in the following integer sequence, called the **Fibonacci sequence:**

1 1 2 3 5 8 13 21 ....

The next number is found by adding up the two numbers before it.

The rule is

$$x_n = x_{n-1} + x_{n-2}$$

where

x_n = term number n

x_{n-1} = previous term $n - 1$

x_{n-2} = the term before the previous term $n - 2$

For example, the 20th term = 19th term + 18th term.

Using these ideas

The 9th term of 1 1 2 3 5 8 13 21 ....

is

$x_9 = x_8 + x_7$ or next term = sum of the previous two terms

$x_9 = 21 + 13 = 34$

Here are the first five terms of the Fibonacci sequence

0 1 1 2 3

This is how we can find the next 5 terms:

RULE: Fibonacci sequence is, the next term in the sequence is the sum of the two previous terms.

0 1 1 2 3 5

2 + 3

0 1 1 2 3 5 8

3 + 5

0 1 1 2 3 5 8 13

5 + 8

0 1 1 2 3 5 8 13 21

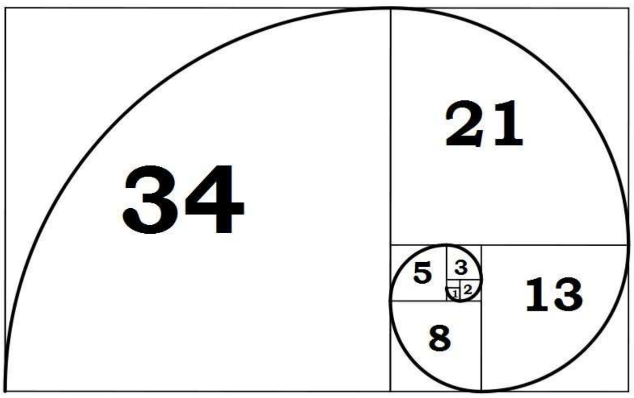
8 + 13

0 1 1 2 3 5 8 13 21 34

13 + 21

Hence the first 10 terms of the Fibonacci sequence are:

0 1 1 2 3 5 8 13 21 34



EXAMPLE 1

Find the first ten terms of the Fibonacci sequence

2 5 7 12 19

The rule to continue a Fibonacci sequence is:

The next term in the sequence is the sum of the two previous terms.

The first 5 terms given are: 2 5 7 12 19

The 6th term = 12 + 19 = 31 ← 6th term = 4th term + 5th term

The 7th term = 19 + 31 = 50 ← 7th term = 5th term + 6th term

The 8th term = 31 + 50 = 81 ← 8th term = 6th term + 7th term

The 9th term = 50 + 81 = 131 ← 9th term = 7th term + 8th term

The 10th term = 81 + 131 = 212 ← 10th term = 8th term + 9th term

Thus, the first 10 terms are:

2 5 7 12 19 31 50 81 131 212

EXERCISE 1:

1. The first fifteen Fibonacci numbers are:

   1 1 2 3 5 8 13 21 34 55 89 144 233 377 610

   What type of number is every third term?

2. Find the first ten terms of the following Fibonacci sequences.

(a)	2	3	5	8	(b)	1	3	4	7
(c)	2	7	9	16	(d)	3	8	11	19
(e)	1	6	7	13	(f)	0	3	3	6
(g)	10	11	21	32	(h)	7	11	18	29
(i)	20	20	40	60	(j)	50	65	115	180

3. A Fibonacci sequence has second term 20 and fifth term 80. Find the first term.

4. The sixth and seventh terms of a Fibonacci sequence are 31 and 50.
   What are the first two terms of the sequence?

5. If you take the first ten terms of any Fibonacci sequence, the sum of those 10 terms is equal to the 7th term multiplied by 11.

   Show that this is true for the following Fibonacci sequence

   4 5 9 14

6. When you find the sum of the first six numbers of a Fibonacci sequence the sum is always four times the fifth number in the sequence.

   Show that this is true for the following Fibonacci sequence

   3 7 10 17

ANSWERS

Exercise 1

1. An even number

2. (a) 13, 21, 34, 55, 89, 144
(b) 11, 18, 29, 47, 76, 123
(c) 25, 41, 66, 107, 173, 280
(d) 30, 49, 79, 128, 207, 335
(e) 20, 33, 53, 86, 139, 225
(f) 9, 15, 21, 36, 57, 93
(g) 53, 85, 138, 223, 361, 584
(h) 47, 76, 123, 199, 322, 521
(i) 100, 160, 260, 420, 680, 1100
(j) 295, 475, 770, 1245, 2015, 3260

3. 10

4. 2,5

5. Sum of the first 10 terms = 660
7th term × 11 = 60 × 11 = 660

6. Sum of the first 6 terms = 108
5th term × 4 = 27 × 4 = 108

EXTENSION: EXAMPLE 2

Find the first six terms of the Fibonacci sequence

$a \qquad b \qquad a + b$

The rule to continue a Fibonacci sequence is:

The next term in the sequence is the sum of the two previous terms.

The first 3 terms given are: $a \qquad b \qquad a + b$

The 4th term $= b + a + b \qquad = a + 2b$ ← 4th term = 2nd term + 3rd term

The 5th term $= a + b + a + 2b \qquad = 2a + 3b$ ← 5th term = 3rd term + 4th term

The 6th term $= a + 2b + 2a + 3b \qquad = 3a + 5b$ ← 6th term = 4th term + 5th term

Thus, the first 6 terms are:

$a \qquad b \qquad a + b \qquad a + 2b \qquad 2a + 3b \qquad 3a + 5b$

EXTENSION: EXAMPLE 3

The first three terms of a Fibonacci sequence are: $x \qquad ny \qquad x + ny$

Show that the sum of the first 6 terms = 4 × the 5th term

The 4th term $= ny + x + ny \qquad = x + 2ny$ ← 4th term = 2nd term + 3rd term

The 5th term $= x + ny + x + 2ny \qquad = 2x + 3ny$ ← 5th term = 3rd term + 4th term

The 6th term $= x + 2ny + 2x + 3ny \qquad = 3x + 5ny$ ← 6th term = 4th term + 5th term

Thus, the sum of the first 6 terms is:

$x + ny + x + ny + x + 2ny + 2x + 3ny + 3x + 5ny$ ← Add the terms in x and the terms in y

$= 8x + 12ny$

$4 \times$ the 5th term $= 4 \times (2x + 3ny)$

$= 8x + 12ny$

Hence the sum of the first 6 terms = 4 × the 5th term (*proved*) ← Finish with a statement

EXERCISE 2:

1. Find the first six terms of the following Fibonacci sequences.

(a)	x	x	$2x$	(b)	x	$2x$	$3x$
(c)	x	$4x$	$5x$	(d)	x	y	$x + y$
(e)	$2x$	$2y$	$2x + 2y$	(f)	$3x$	$5y$	$3x + 5y$

2. The sixth and seventh terms of a Fibonacci sequence are $(12x + 25y)$ and $(20x + 40y)$.

   (a) Write down the first two terms of the sequence.

   (b) Write down the next two terms of this sequence.

3. If you take the first ten terms of any Fibonacci sequence, the sum of those 10 terms is equal to the 7th term multiplied by 11.
   Show this is true algebraically by using the following Fibonacci sequence

   $a \quad b \quad a + b$

4. Here are the first six terms of a Fibonacci sequence.

   1 1 2 3 5 8

   The rule to continue a Fibonacci sequence is, the next term in the sequence is the sum of the two previous terms.

   (a) Find the 9th term of this sequence.

   The first three terms of a different Fibonacci sequence are

   $a \quad b \quad a + b$

   (b) Show that the 6th term of this sequence is $3a + 5b$

   Given that the 3rd term is 7 and the 6th term is 29

   (c) Find the value of a and the value of b.

5. When you find the sum of the first six numbers of a Fibonacci sequence the sum is always four times the fifth number in the sequence.

   Show this is true algebraically by using the following Fibonacci sequence

   $a \quad b \quad a + b$

6. The first three terms of a Fibonacci sequence are $a \quad b \quad a + b$
   The third term is 6 and the fifth term is 17.

   Find the values of a and b.

7. Start with a Fibonacci sequence,

Step 1 Take any four adjacent numbers
Step 2 Square the middle two numbers
Step 3 Find the difference of these squares
Step 4 Multiply the first and fourth numbers together
Step 5 The answers are the same

Show this is true algebraically by using the following Fibonacci sequence

$a \quad b \quad a+b \quad a+2b$

ANSWERS

Exercise 2

1. (a) $3x$, $5x$, $8x$ (b) $5x$, $8x$, $13x$
(c) $9x$, $14x$, $23x$ (d) $x + 2y$, $2x + 3y$, $3x + 5y$
(e) $2x + 4y$, $4x + 6y$, $6x + 10y$ (f) $3x + 10y$, $6x + 15y$, $9x + 25y$

2. (a) $4x$, 5y (b) $32x + 65y$, $52x + 105y$

3. Sum of the first 10 terms = $55a + 88b$
7th term × 11 = $(5a + 11b) \times 11 = 55a + 88b$

4. (a) 55 (b) a, b, $a+b$, $a+2b$, $2a+3b$, $3a+5b$
(c) $a = 3$ and $b = 4$

5. Sum of the first 6 terms = $8a + 12b$
5th term × 4 = $(2a + 3b) \times 4 = 8a + 12b$

6. $a = 1$ and $b = 5$

7. Difference of middle terms squared = 1st term × 4thth term
$(a + b)^2 - b^2 = a \times (a + 2b)$
$a^2 + 2ab + b^2 - b^2 = a^2 + 2ab$
$a^2 + 2ab = a^2 + 2ab$