

Answer ALL questions. Write your answers in the spaces provided.

1. Given

$$2^x \times 4^y = \frac{1}{2\sqrt{2}}$$

express y as a function of x .

(3)

$$4^y = 2^{2y}$$

$$2^x \times 2^{2y} = 2^{x+2y}$$

$$\Rightarrow x+2y = -\frac{3}{2}$$

$$\Rightarrow y = -\frac{3}{4} - \frac{x}{2}$$

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA



2. The speed of a small jet aircraft was measured every 5 seconds, starting from the time it turned onto a runway, until the time when it left the ground.

The results are given in the table below with the time in seconds and the speed in ms^{-1} .

Time (s)	0	5	10	15	20	25
Speed (ms^{-1})	2	5	10	18	28	42

Using all of this information,

- (a) estimate the length of runway used by the jet to take off.

(3)

Given that the jet accelerated smoothly in these 25 seconds,

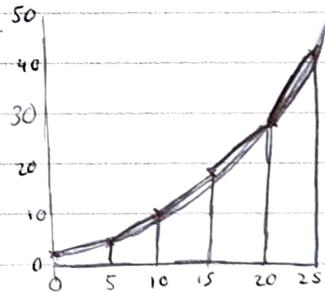
- (b) explain whether your answer to part (a) is an underestimate or an overestimate of the length of runway used by the jet to take off.

(1)

a) Using trapezium rule:

$$\begin{aligned}
 S &\approx \frac{1}{2} \times 5 \times [2 + 42 + 2(5 + 10 + 18 + 28)] \\
 &= \frac{5}{2} \times 166 \\
 &= 415 \text{ m.}
 \end{aligned}$$

b) Since the jet is accelerating, the gradient of the curve is continually increasing. Therefore, the curve ^{connecting two points.} must be ~~above~~ ^{beneath} the line ~~above~~ ^{connecting} two points.



3.

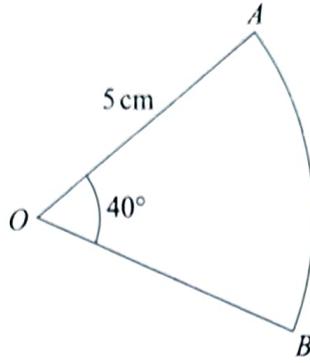


Figure 1

Figure 1 shows a sector AOB of a circle with centre O , radius 5 cm and angle $AOB = 40^\circ$

The attempt of a student to find the area of the sector is shown below.

$$\begin{aligned}
 \text{Area of sector} &= \frac{1}{2} r^2 \theta \\
 &= \frac{1}{2} \times 5^2 \times 40 \\
 &= 500 \text{ cm}^2
 \end{aligned}$$

(a) Explain the error made by this student.

(1)

(b) Write out a correct solution.

(2)

a) The student did not convert 40° to radians.

b) $\frac{1}{2} \times 5^2 \times \frac{40\pi}{360} = \frac{25}{9} \pi \text{ cm}^2$.



4.

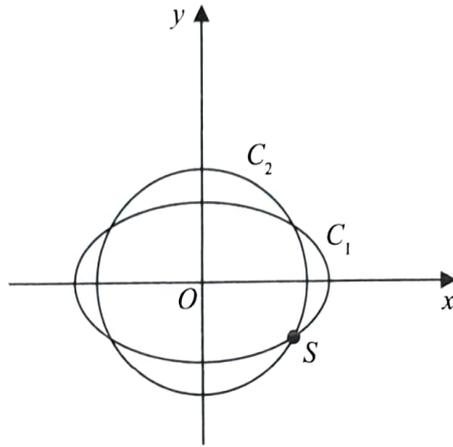


Figure 2

The curve C_1 with parametric equations

$$x = 10\cos t, \quad y = 4\sqrt{2}\sin t, \quad 0 \leq t < 2\pi$$

meets the circle C_2 with equation

$$x^2 + y^2 = 66$$

at four distinct points as shown in Figure 2.

Given that one of these points, S , lies in the 4th quadrant, find the Cartesian coordinates of S . (6)

$$(10\cos t)^2 + (4\sqrt{2}\sin t)^2$$

$$= 100\cos^2 t + 32\sin^2 t$$

$$= 100\cos^2 t + 32(1 - \cos^2 t)$$

$$= 32 + 68\cos^2 t = 66$$

$$\Rightarrow 68\cos^2 t = 34$$

$$\Rightarrow \cos^2 t = \frac{1}{2}$$

$$\Rightarrow \cos t = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \sin t = \frac{1}{\sqrt{2}}$$

$$\Rightarrow x = 10 \cdot \frac{1}{\sqrt{2}} = 5\sqrt{2}, \quad y = -4\sqrt{2} \cdot \frac{1}{\sqrt{2}} = -4.$$

$$S: (5\sqrt{2}, -4).$$



5.

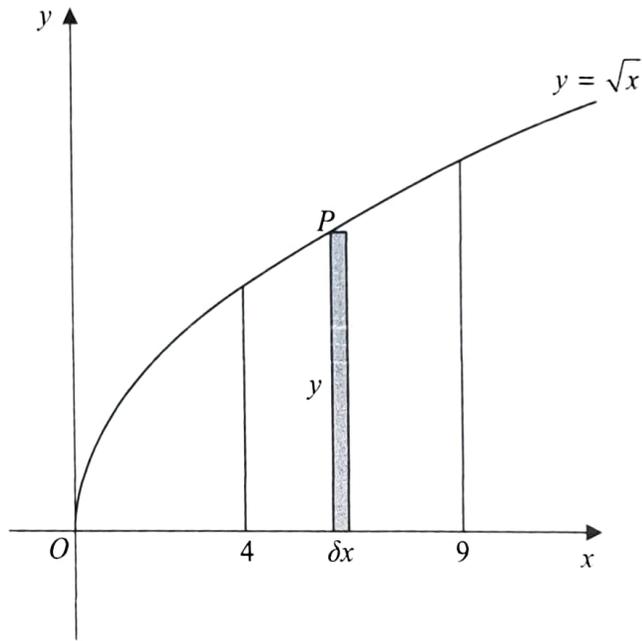


Figure 3

Figure 3 shows a sketch of the curve with equation $y = \sqrt{x}$

The point $P(x, y)$ lies on the curve.

The rectangle, shown shaded on Figure 3, has height y and width δx .

Calculate

$$\lim_{\delta x \rightarrow 0} \sum_{x=4}^9 \sqrt{x} \delta x$$

$$\lim_{\delta x \rightarrow 0} \sum_{x=4}^9 \sqrt{x} \delta x = \int_4^9 \sqrt{x} \, dx \quad (3)$$

$$= \left[\frac{2}{3} x^{3/2} \right]_4^9$$

$$= \left(\frac{2}{3} \times 27 - \frac{2}{3} \times 8 \right)$$

$$= \frac{54}{3} - \frac{16}{3} = \frac{38}{3}$$



6.

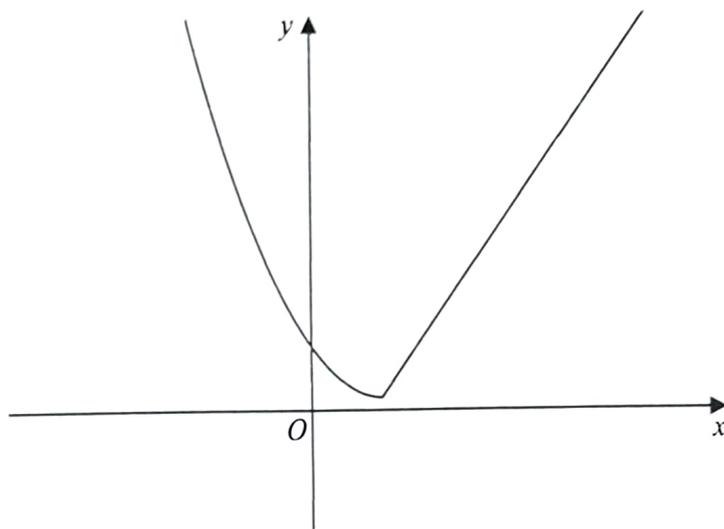


Figure 4

Figure 4 shows a sketch of the graph of $y = g(x)$, where

$$g(x) = \begin{cases} (x-2)^2 + 1 & x \leq 2 \\ 4x - 7 & x > 2 \end{cases}$$

(a) Find the value of $gg(0)$. (2)

(b) Find all values of x for which $g(x) > 28$ (4)

The function h is defined by

$$h(x) = (x-2)^2 + 1 \quad x \leq 2$$

(c) Explain why h has an inverse but g does not. (1)

(d) Solve the equation

$$h^{-1}(x) = -\frac{1}{2} \quad (3)$$

a) $g(0) = 5, \Rightarrow gg(0) = g(5) = 13.$



Question 6 continued

b) For $x \leq 2$:

$$(x-2)^2 + 1 = 28.$$

$$\Rightarrow x-2 = \pm \sqrt{27}$$

$$\Rightarrow x = 2 - \sqrt{27}. \quad (2 + \sqrt{27} \text{ is outside of the domain}).$$

For $x > 2$:

$$4x - 7 = 28$$

$$\Rightarrow 4x = 35$$

$$\Rightarrow x = \frac{35}{4}.$$

$g(x) > 28$ when
 $x < 2 - \sqrt{27}$, or, $x > \frac{35}{4}$.

c) $h(x)$ is a one-one function as it is defined for $x \leq 2$.

$g(x)$ is not a one-one function, so it cannot have an inverse.

d) $h(x) = (x-2)^2 + 1.$

~~h(x) = (x-2)^2 + 1~~

$$h^{-1}(x) = -\frac{1}{2} \Rightarrow h(-\frac{1}{2}) = x.$$

$$= (-\frac{1}{2} - 2)^2 + 1$$

$$= \frac{25}{4} + 1 = \frac{29}{4}.$$



7. A small factory makes bars of soap.

On any day, the total cost to the factory, £ y , of making x bars of soap is modelled to be the sum of two separate elements:

- a fixed cost
- a cost that is proportional to the number of bars of soap that are made that day

(a) Write down a general equation linking y with x , for this model.

(1)

The bars of soap are sold for £2 each.

On a day when 800 bars of soap are made and sold, the factory makes a profit of £500

On a day when 300 bars of soap are made and sold, the factory makes a loss of £80

Using the above information,

(b) show that $y = 0.84x + 428$

(3)

(c) With reference to the model, interpret the significance of the value 0.84 in the equation.

(1)

Assuming that each bar of soap is sold on the day it is made,

(d) find the least number of bars of soap that must be made on any given day for the factory to make a profit that day.

(2)

a) $y = kx + c.$

b) ~~$y = 2x + c$~~ $x = 800 \Rightarrow y = 1600 - 500$
 ~~$y = 2x + c$~~ $= 1100.$

$x = 300 \Rightarrow y = 680.$

$680 = 300k + c$

$1100 = 800k + c.$

$(k=)$ $\frac{680 - c}{300} = \frac{1100 - c}{800}$

$\Rightarrow 5440 - 8c = 3300 - 3c$

$\Rightarrow c = 428.$

$\Rightarrow k = \frac{680 - 428}{300} = 0.84 \Rightarrow y = 0.84x + 428.$



Question 7 continued

c) The cost of making one bar of soap.

d) Let n be the number of bars

Revenue: $2n$.

Cost: $0.84n + 428$

$$2n = 0.84n + 428$$

$$\Rightarrow 1.16n = 428$$

$$\Rightarrow n = 368.97$$

The factory must make 369 bars to make a profit.



8. (i) Find the value of

$$\sum_{r=4}^{\infty} 20 \times \left(\frac{1}{2}\right)^r$$

(3)

(ii) Show that

$$\sum_{n=1}^{48} \log_5 \left(\frac{n+2}{n+1}\right) = 2$$

i) $\sum_{r=1}^{\infty} 20 \times \left(\frac{1}{2}\right)^r = \sum_{r=1}^{\infty} 20 \times \left(\frac{1}{2}\right)^r - \sum_{r=1}^3 20 \times \left(\frac{1}{2}\right)^r$ (3)

$$= \left(\frac{10}{1 - \frac{1}{2}}\right) - \left(10 + 5 + \frac{5}{2}\right)$$
$$= 20 - 17.5 = \frac{5}{2}$$

ii) $\sum_{n=1}^{48} \log_5 \left(\frac{n+2}{n+1}\right) = \log_5 \frac{3}{2} + \log_5 \frac{4}{3} + \dots + \log_5 \frac{50}{49}$

$$= \log_5 \left(\frac{3}{2} \times \frac{4}{3} \times \dots \times \frac{50}{49}\right)$$
$$= \log_5 \frac{50}{2} = \log_5 25 = 2$$



9. A research engineer is testing the effectiveness of the braking system of a car when it is driven in wet conditions.

The engineer measures and records the braking distance, d metres, when the brakes are applied from a speed of $V \text{ km h}^{-1}$.

Graphs of d against V and $\log_{10} d$ against $\log_{10} V$ were plotted.

The results are shown below together with a data point from each graph.

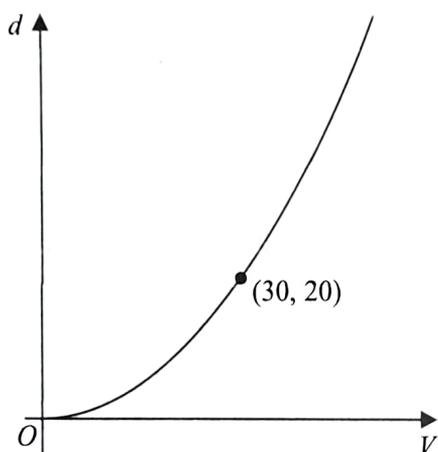


Figure 5

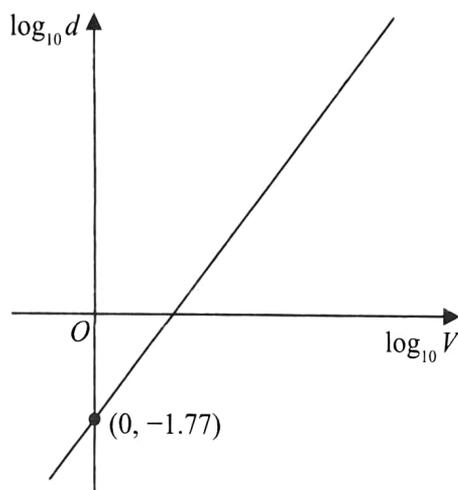


Figure 6

- (a) Explain how Figure 6 would lead the engineer to believe that the braking distance should be modelled by the formula

$$d = kV^n \quad \text{where } k \text{ and } n \text{ are constants}$$

with $k \approx 0.017$

(3)

Using the information given in Figure 5, with $k = 0.017$

- (b) find a complete equation for the model giving the value of n to 3 significant figures.

(3)

Sean is driving this car at 60 km h^{-1} in wet conditions when he notices a large puddle in the road 100 m ahead. It takes him 0.8 seconds to react before applying the brakes.

- (c) Use your formula to find out if Sean will be able to stop before reaching the puddle.

(3)

a) The graph of $\log_{10} d$ against $\log_{10} V$ is linear, i.e.

$$\log_{10} d = n \log_{10} V + \log_{10} k$$



Question 9 continued

which rearranges to

$$d = kV^n$$

$$\log_{10} k = -1.77$$

$$\Rightarrow k = 10^{-1.77} = 0.017$$

b) $d = 20, v = 30.$

$$20 = 0.017 \times 30^n$$

$$\Rightarrow 30^n \approx 1176.47$$

$$\Rightarrow n = \frac{\log 1176.47}{\log 30} = 2.08$$

$$\Rightarrow d = 0.017 v^{2.08}$$

c) $60 \text{ km h}^{-1} = 60000 \text{ m h}^{-1} = 16.6 \text{ m/s}.$

$$v = 60 \Rightarrow d = 0.017 \times 60^{2.08} = 84.92 \text{ m}.$$

$$16.6 \times 0.8 = 13.3 \text{ m}$$

$$84.92 + 13.33 = 98.25 \text{ m}.$$

Yes, he will stop before reaching the puddle.



10.

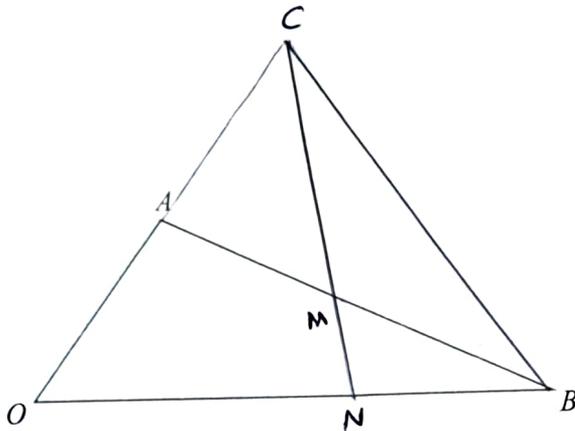


Figure 7

Figure 7 shows a sketch of triangle OAB .

The point C is such that $\vec{OC} = 2\vec{OA}$.

The point M is the midpoint of AB .

The straight line through C and M cuts OB at the point N .

Given $\vec{OA} = \mathbf{a}$ and $\vec{OB} = \mathbf{b}$

(a) Find \vec{CM} in terms of \mathbf{a} and \mathbf{b} (2)

(b) Show that $\vec{ON} = \left(2 - \frac{3}{2}\lambda\right)\mathbf{a} + \frac{1}{2}\lambda\mathbf{b}$, where λ is a scalar constant. (2)

(c) Hence prove that $ON:NB = 2:1$ (2)

$$\begin{aligned} \text{a) } \vec{AM} &= \frac{1}{2}\vec{AB} \Rightarrow \vec{CM} = \vec{CA} + \vec{AM} \\ &= -\vec{AC} + \frac{1}{2}\vec{AB} \\ &= -\vec{OA} + \frac{1}{2}\vec{AB} \end{aligned}$$

$$= -\mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a})$$

$$= -\frac{3}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$$



Question 10 continued

$$b) \vec{ON} = \vec{OC} + \vec{CN}$$

$$= 2\underline{a} + \lambda\left(\frac{3}{2}\underline{a} + \frac{1}{2}\underline{b}\right)$$

$$= \left(2 - \frac{3}{2}\lambda\right)\underline{a} + \frac{1}{2}\lambda\underline{b}$$

c) \vec{ON} does not move in the direction of \underline{a} .

So,

$$2 - \frac{3}{2}\lambda = 0.$$

$$\Rightarrow \lambda = \frac{4}{3}$$

$$\Rightarrow \vec{ON} = \frac{2}{3}\underline{b}.$$

$$\vec{OB} = \underline{b}.$$

$$\Rightarrow \vec{NB} = \frac{1}{3}\underline{b}$$

$$\Rightarrow ON : NB = \frac{2}{3} : \frac{1}{3} = 2 : 1$$



11.

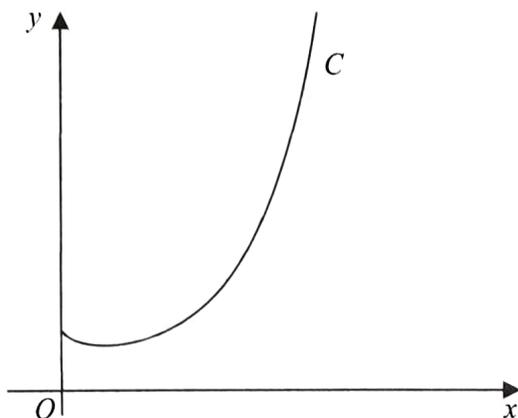


Figure 8

Figure 8 shows a sketch of the curve C with equation $y = x^x$, $x > 0$

(a) Find, by firstly taking logarithms, the x coordinate of the turning point of C .

(Solutions based entirely on graphical or numerical methods are not acceptable.)

(5)

The point $P(\alpha, 2)$ lies on C .

(b) Show that $1.5 < \alpha < 1.6$

(2)

A possible iteration formula that could be used in an attempt to find α is

$$x_{n+1} = 2x_n^{1-x_n}$$

Using this formula with $x_1 = 1.5$

(c) find x_4 to 3 decimal places,

(2)

(d) describe the long-term behaviour of x_n

(2)

$$a) \quad y = x^x \Rightarrow \ln y = x \ln x.$$

$$f = x \quad g = \ln x \\ f' = 1 \quad g' = \frac{1}{x}$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = 1 + \ln x$$

$$y, x \neq 0.$$

~~$$\frac{dy}{dx} = 0 \Rightarrow \ln x = -1$$~~

~~$$\Rightarrow x = e^{-1}$$~~

$$\frac{dy}{dx} = 0 \Rightarrow \ln x = -1.$$

$$\Rightarrow x = e^{-1}$$

~~Handwritten scribbles~~


Question 11 continued

b) $f(1.5) = 1.5^{1.5} = 1.837$

$f(1.6) = 1.6^{1.6} = 2.121$

$f(1.5) < 2$, $f(1.6) > 2$ and the function is continuous,
so the solution $f(x) = 2$ has a solution
 $1.5 < x < 1.6$.

c) $x_2 = 2x_1^{1-x_1}$

$x_1 = 1.5 \Rightarrow x_2 = 1.633$

$\Rightarrow x_3 = 1.466$

$\Rightarrow x_4 = 1.673$

d) x_n is divergent, it oscillates between
1 and 2.



12. (a) Prove

$$\frac{\cos 3\theta}{\sin \theta} + \frac{\sin 3\theta}{\cos \theta} \equiv 2 \cot 2\theta \quad \theta \neq (90n)^\circ, n \in \mathbb{Z} \quad (4)$$

(b) Hence solve, for $90^\circ < \theta < 180^\circ$, the equation

$$\frac{\cos 3\theta}{\sin \theta} + \frac{\sin 3\theta}{\cos \theta} = 4$$

giving any solutions to one decimal place.

$$\begin{aligned} \text{a) } \frac{\cos 3\theta \cos \theta + \sin 3\theta \sin \theta}{\sin \theta \cos \theta} &= \frac{\cos(3\theta - \theta)}{\sin \theta \cos \theta} \\ &= \frac{\cos 2\theta}{\frac{1}{2} \sin 2\theta} \\ &= 2 \cot 2\theta. \end{aligned} \quad (3)$$

$$\begin{aligned} \text{b) } 2 \cot 2\theta &= 4 \\ \Rightarrow \tan 2\theta &= \frac{1}{2}. \end{aligned}$$

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{1}{2}.$$

$$\Rightarrow 2 \tan \theta = \frac{1}{2} - \frac{1}{2} \tan^2 \theta.$$

$$\Rightarrow \tan^2 \theta + 4 \tan \theta - 1 = 0.$$

$$\Rightarrow \tan \theta = \frac{-4 \pm \sqrt{16 + 4}}{2} = -2 \pm \sqrt{5}.$$

$$90^\circ < \theta < 180^\circ \Rightarrow \tan \theta < 0.$$

$$\begin{aligned} \Rightarrow \tan \theta &= -2 - \sqrt{5} \\ \Rightarrow \theta &= 103.3^\circ. \end{aligned}$$



13.

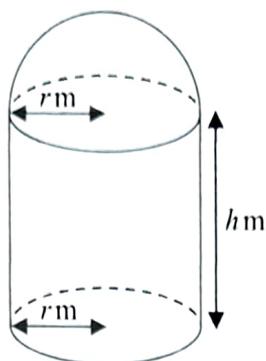


Figure 9

[A sphere of radius r has volume $\frac{4}{3}\pi r^3$ and surface area $4\pi r^2$]

A manufacturer produces a storage tank.

The tank is modelled in the shape of a hollow circular cylinder closed at one end with a hemispherical shell at the other end as shown in Figure 9.

The walls of the tank are assumed to have negligible thickness.

The cylinder has radius r metres and height h metres and the hemisphere has radius r metres.

The volume of the tank is 6 m^3 .

(a) Show that, according to the model, the surface area of the tank, in m^2 , is given by

$$\frac{12}{r} + \frac{5}{3}\pi r^2 \quad (4)$$

The manufacturer needs to minimise the surface area of the tank.

(b) Use calculus to find the radius of the tank for which the surface area is a minimum. (4)

(c) Calculate the minimum surface area of the tank, giving your answer to the nearest integer. (2)

$$\begin{aligned} \text{a) Volume: } h\pi r^2 + \frac{2}{3}\pi r^3 &= 6 = (h + \frac{2}{3}r)\pi r^2 \\ \Rightarrow \text{~~h} &= \frac{6}{\pi r^2} - \frac{2}{3}r. \end{aligned}~~$$

$$\begin{aligned} \text{Area: } \underbrace{2h\pi r}_{\substack{\text{curved} \\ \text{side}}} + \underbrace{2\pi r^2}_{\text{hemisphere}} + \underbrace{\pi r^2}_{\text{base}} &= 3\pi r^2 + 2\pi r \left(\frac{6}{\pi r^2} - \frac{2}{3}r \right) \\ &= 3\pi r^2 + \frac{12}{r} - \frac{4}{3}\pi r^2 \\ &= \frac{12}{r} + \frac{5}{3}\pi r^2. \end{aligned}$$



Question 13 continued

b) Area: $A = \frac{12}{r} + \frac{5}{3} \pi r^2$

$$\Rightarrow \frac{dA}{dr} = -\frac{12}{r^2} + \frac{10}{3} \pi r = 0$$

$$\Rightarrow \frac{10}{3} \pi r^2 - 12 = 0$$

$$\Rightarrow r^3 = \frac{12}{\frac{10}{3} \pi} = \frac{3.6}{\pi}$$

$$\Rightarrow r = \sqrt[3]{\frac{18}{5\pi}} = 1.046 \text{ m.}$$

c) $A = \frac{12}{1.046} + \frac{5}{3} \pi (1.046)^2 = 17.20 \text{ m}^2.$
 $\Rightarrow 17 \text{ m}^2.$



14. (a) Use the substitution $u = 4 - \sqrt{h}$ to show that

$$\int \frac{dh}{4 - \sqrt{h}} = -8 \ln|4 - \sqrt{h}| - 2\sqrt{h} + k$$

where k is a constant

(6)

A team of scientists is studying a species of slow growing tree.

The rate of change in height of a tree in this species is modelled by the differential equation

$$\frac{dh}{dt} = \frac{t^{0.25}(4 - \sqrt{h})}{20}$$

where h is the height in metres and t is the time, measured in years, after the tree is planted.

(b) Find, according to the model, the range in heights of trees in this species.

(2)

One of these trees is one metre high when it is first planted.

According to the model,

(c) calculate the time this tree would take to reach a height of 12 metres, giving your answer to 3 significant figures.

(7)

$$a) \quad u = 4 - \sqrt{h} \quad \Rightarrow \quad \frac{du}{dh} = -\frac{1}{2} h^{-1/2} = -\frac{1}{2\sqrt{h}}$$

$$\Rightarrow du = -\frac{1}{2\sqrt{h}} dh \quad \Rightarrow \quad dh = -2\sqrt{h} du$$

$$\Rightarrow \int \frac{dh}{4 - \sqrt{h}} = \int \frac{-2\sqrt{h} du}{u} = \int \frac{2(u-4)}{u} du$$

$$= \int \left(2 - \frac{8}{u}\right) du,$$

$$= 2u - 8 \ln u + c$$

$$= 2(4 - \sqrt{h}) - 8 \ln(4 - \sqrt{h}) + c$$

$$= 8 - 2\sqrt{h} - 8 \ln|4 - \sqrt{h}| + c$$

$$= -8 \ln|4 - \sqrt{h}| - 2\sqrt{h} + c$$



Question 14 continued

$$b) \quad \frac{dh}{dt} = 0 \Rightarrow t=0, \quad 4 - \sqrt{h} = 0.$$

$$\Rightarrow h = 16.$$

$$\Rightarrow 0 \leq h \leq 16$$

$$c) \quad \frac{dh}{dt} = \frac{t^{0.25}(4 - \sqrt{h})}{20}$$

$$\Rightarrow \int \frac{dh}{4 - \sqrt{h}} = \frac{1}{20} \int t^{0.25} dt$$

$$\Rightarrow -8 \ln |4 - \sqrt{h}| - 2\sqrt{h} = \frac{1}{25} t^{1.25} + C.$$

$$t=0 \Rightarrow h=1.$$

$$-8 \ln 3 - 2 = C.$$

$$\Rightarrow -8 \ln |4 - \sqrt{h}| - 2\sqrt{h} = \frac{1}{25} t^{1.25} - 8 \ln 3 - 2.$$

$$h=12:$$

$$-8 \ln (4 - \sqrt{12}) - 2\sqrt{12} = \frac{1}{25} t^{1.25} - 8 \ln 3 - 2.$$

$$\Rightarrow t = 75.15 \approx 75.2 \text{ years.}$$

